Integral Equations

Integral equations arise in a number of areas of mathematical physics and often characterise the solution of inverse problems. Integral equations can often be solved analytically, but they are often otherwise solved by numerical methods[1]; replacing the integral equation by an analogous linear system of equations[2]. In particular, partial differential equations[3] can often be reformulated as an integral equation, forming the basis of methods such as the boundary element method[4].

In their simplest form, integral equations are equations in one variable (say t) that involve an integral[5] over a domain of another variable (s) of the product of a kernel function *K(s,t)* and another (unknown) function (*f(s)*). The purpose of a solution method is to determine the unknown function *f*. If the limit(s) on the integration domain are fixed then it is said to be a Fredholm Equation. If the limit(s) on the integration domain are not fixed then it is said to be a Volterra Equation.

In this document the basic form and terminology of integral equations is introduced. The shorthand operator notation and the analogy with linear systems of equations is outlined.

Fredholm Equations

Fredholm equations occur in two forms: *Fredholm equations of the first kind* and *Fredholm equations of the second kind*.

Fredholm Equations of the First Kind have the form

$$g(t) = \int_a^b K(s,t)f(s)dt \tag{1}$$

where the function g is unknown at the outset. $K(s,t)$is termed the kernel function.

Fredholm Equations of the Second Kind have the form

$$f(t) + \lambda \int_a^b K(s,t)f(s)dt = g(t). \tag{2}$$

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Volterra Equations

Volterra equations (of the first and second kind) are defined similarly, except the domain of integration is dependent on the variable t. For example the second kindequation has the form

$$f(t) = \lambda \int_a^t K(s,t) f(s) dt + g(t).$$

Operator Notation

Operator notation is often used in the analysis and communication of integral equations. For example the equation (1) can be written as follows:

$$g = Kf. \tag{1*}$$

Equation (2) can be written as

$$f + \lambda K f = g. \tag{2*}$$

Operator notation is similarly applicable to the Volterra equations.

Analogy with Linear Systems

In the numerical solution of integral equations, for example by the method of collocation[6], the integral is replaced by a linear system of equation and it is useful to explore the relative terminologies of these forms of equation. For example, equations (2/2*) is analogous to the matrix equation

$$\underline{f} + \lambda\, K\, \underline{f} = \underline{g}.$$

For the homogeneous form of the equation ($g = 0, \underline{g} = \underline{0}$),

$$f + \lambda K f = 0 \text{ and its analogy } \underline{f} + \lambda\, K\, \underline{f} = \underline{0}.$$

In analogy to the algebraic eigenvalue problem[7], if the kernel function is bounded, the homogeneous integral equation has a set of solutions λ, and the values of λ, λ_1,λ_2, ... are termed the eigenvalues. The corresponding solution functions f , f_1,f_2,.... are termed the eigenfunctions.

[6] Solution of Integral Equations by Collocation

[7] Matrix Eigenvalues and Eigenvectors